Examining Hyponormality of Sequence of Toeplitz Operators of Symbols on Bergman Spaces

Huwaida Elsiddig Haroun Mohammed
Department of Mathematics, University College of Umlj, University of Tabuk, Tabuk 48322,Saudi Arabia

Abstract: For a sequence of φ_{ij} in $L^{\infty}(\partial D)$, let $\sum_{i,j=1}^{\infty} P_{ij} = \sum_{i=1}^{\infty} f_i + \sum_{j=1}^{\infty} \overline{g_j}$Where f_i and g_j are two corresponding sequences in H^{∞} . In this paper, it is shown that the Toeplitz operators $T_{\varphi_{ij}}$ are hyponormals if and only if $g_r = c + T_{\overline{h}_r} f_r$ for some constant c and some functions h_r in $H^{\infty}(\partial D)$ with $\|h_r\| \leq 1$ for any $r \geq 1$.

And, for a bounded analytic sequence $\varphi_{ij}\ on\ D$, we show that a sequence of a Toeplitz operator $T_{\varphi_{ij}}$are analytic if $\varphi_{ij}\ is\ in\ H^{\infty}$. These are easily seen to be subnormal:

$$T_{\varphi_{ij}} h_r = P_{ij} \varphi_{ij} h_r = L_{\varphi_{ij}} h_r\ for\ h_r\ in\ H^{\infty}$$

Where $L_{\varphi_{ij}}$ are normal operator of multiplication by φ_{ij} in $L^{\infty}(\partial D)$.

Lastly, for $\varphi_{ij} \in L^{\infty}$ we show that, the sequence of Toeplitz operator $T_{\varphi_{ij}}$ on the Bergman space with symbol φ_{ij} given by:

$$T_{\varphi_{ij}} f_r := P_{ij}(\varphi_{ij} h_r)(f_r \in A^2).$$

are analytic if $\varphi_{ij} \in H^{\infty}$.

Introduction

For φ_{ij} in $L^{\infty}(\partial D)$, the sequence of the Toeplitz operators $T_{\varphi_{ij}}$ for $i,j \geq 1$, are the operators on $H^2$of the unit disc D given by $T_{\varphi_{ij}} u = P_{ij} \varphi_{ij} u$, for $i,j \geq 1$ where P_{ij} are the orthogonal projections of $L^2(\partial D)$ on H^2 . An operator A is called hyponormal if $A^* A - A A^* \geq 0$. Brown and Halmos began the systematic study of the algebraic properties of Toeplitz operators and showed in [1] , that $T_{\varphi_{ij}}$ are normal if and only if $\varphi_{ij} = \alpha_i + \beta_j P_{ij}$ for any $i,j \geq 1$ where α_i, β_j are complex numbers and the sequence P_i are real-valued functions in L^{∞} .

An operator A is called hyponormal if $A^*A - AA^* \geq 0$ .Also An operator S on a Hilbert space $\mathcal{H}$ is subnormal if there is a normal operator N on $K \supset \mathcal{H}$ such that $\mathcal{H}$ is invariant for N and. $N \setminus_x = S$.

Problem 5of Halmos's 1970 lectures "Ten problems in Hilbert space Is every subnormal Toeplitz operator normal or analytic?

All progress on this question has begun with the study of self-commutator of $T_{\varphi_{ij}}$. A subnormal operator S hyponormal, that is, its self-commutator $S^*S - SS^*$ is positive. Also we look at Toeplitz operators on Hardy space $H^2(\pi)$ on the unit circle, that is, operators obtained by compressing multiplication operators on L^2 spaces on Hilbert spaces. And on $A^2(D)$ the Bergman space on the unit disk D , a Toeplitz operators with the symbols of the form:

$\varphi_{ij} \equiv \alpha z^n + \beta z^m + \gamma\ z^{-p} + \delta\ z^{-q}$,Where $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ and $m,n,p,q \in \mathbb{Z}_+$, $m<n$ and $p<q$.

$By\ letting\ T_{\varphi_{ij}} act\ on\ vectors\ of\ the\ form$

$z^k + cz^l + dz^r (k, < l < r)$.

Methodology

In this study, we use a mathematical case study of Toeplitz operators with symbols that exhibit hyponormality in Bergman space.

Objectives:

1) Discuss problem 5 of Holomos's 1970 lectures " Ten Propblem in Hilbert space: is every subnormal Toeplitz operator either normal or analytic?" All progress on this question has begun with the study of self-commutator of $T_{\varphi_{ij}}$. A subnormal operator S hyponormal, that is, its self-commutator $S^*S - SS^*$ is positive.

2) we look at Toeplitz operators on Hardy space $H^2(\pi)$ on the unit circle, that is, operators obtained by compressing multiplication operators on L^2 spaces on Hilbert spaces. And on $A^2(D)$ the Bergman space on the unit disk D , a Toeplitz operators with the symbols of the form:

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$z^k + cz^l + dz^r (k, < l < r)$.

Literature review

Brown and Halmos began the systematic study of algebraic properties of Toeplitz operators and showed,[3,p.98], that T_φ is normal if and only if $\varphi = \alpha + \beta\rho$ where α and β are complex numbers and ρ is a real valued function in L^∞. Properties of hyponormality of Toeplitz operators have played an important role in work on Halmos;s problem 5, [7],

"is every subnormal Toeplitz operator either normal or analytic?"

The question is natural because the two classes, the normal and analytic Toeplitz operators are fairly well understood and are obviously subnormal.The normal Toeplitz operators were characterized by Brown and Halmos in 1964.

In 1984, C.Cowen and J. Jong answered this question in the negative [3].A long the way, C.Cowen obtained a characterization of hyponormality

For Toeplitz operators, as following [3]: If $\varphi \epsilon L^{\infty}, \varphi = \bar{f} + g \ (f, g \ \epsilon H^2)$, then T_{φ} is hyponormal $\Leftrightarrow$ for some $c \epsilon C, h \ \epsilon H^{\infty}, and \ \|h\|_{\infty} \le 1.$

T.Nakazi and K. Takahashi [11] later found an alternative description:

For

$\varphi \ \epsilon L^{\infty}, let \ \varepsilon(\varphi) := \{k \ \epsilon \ H^{\infty}: \|k\|_{\infty} \le 1 \ and \ \varphi - k\bar{\varphi} \epsilon H^{\infty}\} ; then \ T_{\varphi}$ is hyponormal $\Leftrightarrow$ $\varepsilon(\varphi) \neq \varphi$.

(for ageneralization of Cowen'sresult, see[7]).

Aim:

This paper aims to investigate the conditions under which sequence of Toeplitz operators with symbols exhibit hyponormality on Bergman space.

1. Sequence of Toeplitz operators on Hardy spaces:

Let φ_{ij} be in $L^{\infty}(\partial D)$, the sequence of Toeplitz operators $T_{\varphi_{ij}}$ are operators on H^2 of the unit disk D given by $T_{\varphi_{ij}} \ u = p_{ij} \ \varphi_{ij} \ u$ where p_{ij} are the orthogonal projections of $L^2(\partial D)$ onto H^2 .

$T_{\varphi_{ij}}$ are normal if and only if $\varphi_{ij} = \alpha_i + \beta_j \ \rho_{ij}$ where α_i and β_j are complex numbers and ρ_{ij} are real- valued functions in L^{∞} .

Let ψ is in L^{∞} , the Hankel operator H_{ψ} is operator on H^2 given by

$$H_{\psi} \, u = J(\, I - p)(\, \psi \, u) \ ,$$

Where J is unitary operator from H^2 onto $H^{2^{\perp}}$, $J\left(e^{-in\theta}\right) = e^{i(n-1)\theta}$.

Another way: Let v^* be a function define by $v^*\left(e^{i\theta}\right) = \overline{v\left(e^{-i\theta}\right)}$, then H_{ψ} is the operator on H^2 defined by

$\left\langle 2uv, \overline{\psi} \right\rangle = \left\langle H_{\psi} \, u, v^* \right\rangle$, for all $v \in H^{\infty}$ (1).

Necessary facts about Hankel operators include.

(i) $H_{\psi_1} = H_{\psi_2}$ if and only if $(I - p)\psi_1 = (I - p)\psi_2$.

(ii) $\| H_{\psi} \| = \inf \{ \| \varphi \|_{\infty} : (I - p)\psi = (\, I - p)\varphi \}$

(iii) $H^*_{\psi} = H_{\psi^*}$.

(iv) Either H_{ψ} is one – to – one or ker $\left(H_{\psi}\right) = \chi H^2$ where χ is an inner function. The closure of the range of H_{ψ} is H^2 in the former case and $\left(\chi^* H^2\right)^{\perp}$ in the latter.

(v) $H_{\psi} U = U^{*} H_{\psi}$ Where U is unilateral (forward) shift on H^2

Theorem 1: If φ_j are in $L^{\infty}(\partial D)$, where $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varphi_{ij} = \sum_{i=1}^{\infty} f_i + \sum_{j=1}^{\infty} \bar{g}_j$ for f_i and g_j in H^{∞} , then $T_{\varphi_{ij}}$ are hyponormals if and only if

$$\sum_{r=1}^{\infty} g_r = c + T_{\bar{h}_r} \sum_{r=1}^{\infty} f_r$$

For some constant c and some functions h_r in $H^{\infty}(\partial D)$ with $\|h_r\| \leq 1$.

Proof. Let $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varphi_{ij} = \sum_{i=1}^{\infty} f_i + \sum_{j=1}^{\infty} \bar{g}_j$ where f_i and g_j are in H^2 . For every polynomial p in H^2 ,

$$\langle \left(T^{*}{}_{\varphi_{ij}} T_{\varphi_{ij}} - T_{\varphi_{ij}} T^{*}{}_{\varphi_{ij}}\right)(p), p\rangle = \langle T_{\varphi_{ij}}, pT_{\varphi_{ij}}\rangle - \langle T^{*}{}_{\varphi_{ij}} p, T^{*}{}_{\varphi_{ij}}\rangle$$
$$= \langle f_i p + P\bar{g}_j p, f_i p + P\bar{g}_j p\rangle - \langle p\bar{f}_i p + g_j p, p\bar{f}_i p + g_j p\rangle$$
$$= \langle \bar{f}_i p, \bar{f}_i P\rangle - \langle P\bar{f}_i P, P\bar{f}_i P\rangle - \langle \bar{g}_j p, \bar{g}_j p\rangle + \langle P\bar{g}_j p, P\bar{g}_j P\rangle$$
$$= \langle \bar{f}_i P, (I-p)\bar{f}_i p\rangle - \langle \bar{g}_j P, (I-p)\bar{g}_j p\rangle$$
$$= \langle (1-p)\bar{f}_i p, (I-p)\bar{f}_i \bar{p}\rangle - \langle (I-p)\bar{g}_j p, (I-p)\bar{g}_j p\rangle$$
$$= \left\|H_{\bar{g}_j} p\right\|^2 - \left\|H_{\bar{F}_i} p\right\|^2$$

Since the polynomial are dense in H^2 and since the Hankel and Toeplitz operators involved are bounded , then $T_{\varphi_{ij}}$ is hyponormal if and only if for all u in H^2

$$\left\|H_{\bar{g}_j} u\right\| \leq \left\|H_{\bar{f}_i} u\right\| . \qquad (2)$$

Let K denote the closure of the range of $H_{\bar{f}_i}$, and let S denote the compression of U to K . Since k is invariant for U^{*} , the operator S^{*} is the restriction of U^{*} to K .

Suppose first $T_{\varphi_{ij}}$ is hyponormal . Define an operator A on the range of $H_{\bar{f}_i}$ by

$$A\left(H_{\bar{f}_i} u\right) = H_{\bar{g}_j} u .$$

If $H_{\bar{f}_i} u_1 = H_{\bar{f}_i} u_2$, so that $H_{\bar{f}_i}(u_1 - u_2) = 0$ then the inequality (2) implies that $H_{\bar{g}_j}(u_1 - u_2) = 0$ too and it follows that A is well defined . Moreover , inequality (2) implies that $\|A\| \leq 1$ so A has an extension to k , which will also be denoted A with the same norm .

Now by the intertwining formula for Hankel operators and the fact that k is invariant for U^*, we have

$H_{\bar{g}_j} U = AH_{\bar{f}_i} U = AU^* H_{\bar{f}_i} = AS^* H_{\bar{f}_i}$

And also

$H_{\bar{g}_j} U = U^* H_{\bar{g}_j} = U^* AH_{\bar{f}_i} = S^* AH_{\bar{f}_i}$.

Since the range of $H_{\bar{f}_i}$ is dense in k, we find that $AS^* = S^* A$ on k , or taking adjoints , that

$S A^* = A^* S$.

By the usual theory of the unilateral shift $k = H^2$ there is function k in $H^\infty(\partial D)$ with $\| k \|_\infty = \| A^* \| = \| A \|$ such that A^* is the compression to k of T_k . Since k is invariant for $T_k^* = T_{\bar{k}}$ this means that A^* is the restriction of $T_{\bar{k}}$ to K and

$$H_{\bar{g}_j} = T_{\bar{k}} H_{\bar{f}_i} \qquad (3)$$

Conversely, if equation (3) holds for some k in $H^\infty(\partial D)$ with $\| k \|_\infty \leq 1$, then clearly inequality (2) holds for all u , and $T_{\varphi_{ij}}$ is hyponormal . By using the formulation (1) , equation (3) holds if and only if for all H^∞ functions u, v ,

$$\langle zuv, g_j \rangle = \langle H_{\bar{g}_j} u, v^* \rangle = \langle T_{\bar{k}} H_{\bar{f}_i} u, v^* \rangle$$

$$= \langle H_{\bar{f}_i} u, kv^* \rangle = \langle zuk^* v f_i \rangle \qquad (\acute{3})$$

$$= \langle zuv, \overline{k^*} f_i \rangle = \langle zuv, T_{\overline{k^*}} f_i \rangle$$

Since the closed span of $\{ zuv : u, v \in H^\infty \}$ is zH^2 this means that equation (3) holds if and only if

$g_j = c + T_{\bar{h}} f_i$

For $h = k^*$ (Note that $\| k \|_\infty = \| k^* \|_\infty$)

In the cases for which $T_{\varphi_{ij}}$ are normal, h is a constant of modulus 1 and in the cases for which $T_{\varphi_{ij}}$ are known to be subnormal but not normal h is a constant of modulus less than 1 .

Remark 2: The functions h_r that relate. f_i and g_j are unique .

Proof. Suppose h_1 and h_2 are in H^∞ and $c_1 + T_{\bar{h}_1} f_i = g_j = c_2 + T_{\bar{h}_2} f_i$ .This is possible if and only if

$T_{\bar{z}} T_{\bar{h}_1} f_i = T_{\bar{z}} T_{\bar{h}_2} f_i$,

That is , if and only if

$T_{\overline{zh_1 - zh_2}} f_i = 0$

Thus , f_i must be in $\left(z\chi H^2\right)^{\perp}$ where χ is the inner factor of $h_1 - h_2$. If f_i is not in any such subspace , the corresponding function h must be unique for every g_j.

On the other hand , if χ is an inner function such that f_i is in $\left(z\chi H^2\right)^{\perp}$ and $c_1 + T_{\overline{h_1}} f_i = g_j$, then for any h_3 in H^{∞} and $h_2 = h_1 + z\chi h_3$, if follows that $g_j = c_2 + T_{\overline{h_2}} f_i$ for some constant c_2 .

Definition 3: Let $H = \{v \in H^{\infty} : v = (0) = 0$ and $\| v \|_2 \le 1\}$. For f_i in H^2 and let $G_{f\,i}$ denote the sets of g_j in H^2 such that for every u in H^2 ,

$$\sup_{v_0 \in H} \left| \langle u v_0 , g_j \rangle \right| \le \sup_{v_0 \in H} \left| \langle u v_0 , f_i \rangle \right|$$

If f_i is in H^{∞} and u is in H^2 , then by (1) ,

$$\sup_{v_0 \in H} \left| \langle u v_0 , f_i \rangle \right| = \left\| H_{\overline{f_i}} u \right\|$$

Theorem 4: If f_i and g_j are in H^{∞} , then g_j are in $G_{f\,i}$ if and only if

$$\sum_{j=1}^{\infty} g_j = c + T_{\bar{h}\,j} \sum_{i=1}^{\infty} f_i \ .$$

For some constant c and some functions h_j in $H^{\infty}(\partial D)$ with $\| h_j \|_{\infty} \le 1$.

Proof . Let $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varphi_{ij} = \sum_{i=1}^{\infty} f_i + \sum_{j=1}^{\infty} \bar{g}_j$. Where f_i and g_j are in H^{∞} from [6] , for every polynomial p_j in H^2 , $\langle \left(T^*{}_{\varphi_{ij}} T_{\varphi_{ij}} - T_{\varphi_{ij}} T^*{}_{\varphi_{ij}}\right)(p), p \rangle =$.

$$\left\| H_{\overline{f_i}} P \right\|^2 - \left\| H_{\overline{g_j}} P \right\|^2$$

Since the polynomials are dense in H^2 and since the Hankel and Toeplitz operators involved are bounded , we see that $T_{\varphi_{ij}}$ are hypomormal if and only if for all u in H^2 ,

$$\left\| H_{\bar{g}_j} u \right\| \le \| H_{\bar{f}_i} u \|$$

Let k denote the closure of the range of $H_{\bar{f}}$ and let S denote the compression of U to K . Since K is invariant for U^*, the operator S^* is the restriction of U^* to K .

Suppose first that $T_{\varphi_{ij}}$ is hyponormal . Define an operator A on the range of $H_{\bar{f}_i}$ by

$$A\left(H_{\bar{f}_i} u\right) = H_{g_j} u \quad .$$

If $H_{\bar{f}_i} u_1 = H_{\bar{f}_i} u_2$, so that $H_{\bar{f}_i}(u_1 - u_2) = 0$, then the inequality (2) implies that $H_{\bar{g}_j}(u_1 - u_2) = 0$ too and it follows that A is well defined . Moreover (2) implies $\| A\|\leq 1$ so A has an extension to K , which will also be denoted A, with the same norm .

Now by the intertwining formula for Hankel operators and the fact that K is invariant for U^*, we have

$$H_{g_j}U = AH_{\bar{f}_i}U = AU^*H_{\bar{f}_i} = AS^*H_{\bar{f}_i}$$

and also

$H_{\bar{g}_i}U = U^*H_{\bar{g}_i} = U^*AH_{\bar{f}_i} = S^*AH_{\bar{f}_i}$.

Since the range of $H_{\bar{f}}$ is dense in K , we find that $AS^* = S^*A$ on K, or taking adjoins , that $SA^* = A^*S$.

By [5] the usual theory of the unilateral shift if $K = H^2$, there is a function k in $H^2(\partial D)$ with $\| K\|_\infty = \| A^*\| = \| A\|$ such that A^* is the compression to K of T_k. Since K is invariant for $T_k^* = T_{\bar{k}}$. This means that A is the restriction of $T_{\bar{k}}$ to K and

$H_{\bar{g}_j} = T_{\bar{k}}H_{\bar{f}_i}$,

is holds if and only if for all H^∞ functions u, v, satisfy $(\acute{3})$.

By using relation (1) $\langle zuv, T_{\bar{k}^*}\rangle = T_{\bar{k}}H_{\bar{f}} = H_{\bar{g}}$;

Now by using the definition there exist $H = \{v \in H^\infty : v(0) = 0 \text{ and } \| v\|_2 \leq 1\}$.

If f_i are in H^∞ and u is in H^2 , then by relation (1)

$$sup_{v\in H}|\langle uv, f_i\rangle| = \|H_{\bar{f}}u\|$$

$$\sup_{v\in H} |\langle uv, f_i\rangle| = \|H_{\bar{f}_i}u\|$$

Thus (2) holds then g is in G_{f_i} definition such that for every u in H^2,

$sup_{v\in H}|\langle uv, g_j\rangle| \leq sup_{v\in H}|\langle uv, f_i\rangle|$.

Thus T_{ij} are hyponormal , then from Theorem

$g_j = c + T_h f_i$ for $h = k^*\left(\| k\|_\infty = \| k^*\|_\infty\right)$.

Corollary 5: For f_j in H^2 , the following hold .

(i) .f_i are in G_{f_i}.

(ii) If g_j are in G_{f_i} , then $g_j + \lambda$ is in G_{f_i} for all complex numbers λ .

(iii) G_{f_i} are balanced and convex , that is , if g_1 and g_2 are in G_{f_i} and $|s_1| + |s_2| \leq 1$, then $s_1 g_1 + s_2 g_2$ is also in G_{f_i}.

(iv) G_{f_i} are weakly closed .

(v) $T_{\bar{\chi}}\, G_{f_i} \subset G_{f_i}$ for every inner function χ .

Conversely, If G is a set that satisfies properties (i) to (v) , then $G \supset G_{f_i}$.

2. Sequence of Subnormal Toplitz operators:

Theorem 6:The Toeplitz operators $T_{\varphi_{ij}}$ are normal if and only if $\varphi_{ij} = \alpha_i + \beta_j \rho_i$, for any $i,j \geq 1$ $where$ α_i and β_j are complex numbers and the sequence ρ_i are real-valued functions $in\ L^\infty$, is real valued.

Theorem 7: If φ_{ij} $is\ in\ \chi\ and\ \bar{\chi}$, $where\ \chi\ is\ an\ innr\ function$, $then\ T_{\varphi_{ij}}$ are subnormal if and only if they are normal or analytic.

Corollary ( ): If $T_{\varphi_{ij}}$ $are\ subnormal\ and\ \varphi_{ij}\ or\ \overline{\varphi_{ij}}$ are of bounded type, then are normal or analytic.

3. Sequence of Toeplitz operators in Bergman space:

Theorem 8: Assume that $T_{\varphi_{ij}}$ are hyponormal on $A^2(D)$, with $\varphi_{ij} \equiv \alpha_i\ z^n + \beta_j z^m + \gamma_i\ z^{-p} + \delta_j\ z^{-q}\ (n>m\,;p<q)$, .Assume also $n - m = q - p$. Then $|\alpha_i|^2 n^2 + |\beta_j|^2 m^2 - |\gamma_i|^2 p^2 - |\delta_j|^2 q^2 \geq 2\left|\overline{\alpha_i}\beta_j mn - \overline{\gamma_i}\delta_j pq\right|$.

Lemma 9 Let φ_{ij} be harmonic and bounded on . Then $T_{\varphi_{ij}}$ are normal if and only if there exists a pair of complex numbers a and b such that $(a\,,b) \neq (0\,,0)$ and $F := a\,\varphi_{ij} + b\,\overline{\varphi_{ij}}$ are constant on D.

The significance:

This research contributes to the ongoing understanding of operator theory by characterizing the hyponormality conditions for further applications in functional analysis and complex analysis.

Conclusion

This paper establishes conditions under which sequence of Toeplitz operators with harmonic and bounded symbols are hyponormal and analytic on Bergman spaces.

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